

A LÉVY-KHINCHIN FORMULA FOR FREE GROUPS

UFFE HAAGERUP AND SØREN KNUDBY

ABSTRACT. We find a Lévy-Khinchin formula for radial functions on free groups. As a corollary we obtain a linear bound on the growth of radial, conditionally negative definite functions on free groups of two or more generators.

1. INTRODUCTION

A function φ defined on a group G is called symmetric, if $\varphi(x) = \varphi(x^{-1})$ for every $x \in G$. It is an elementary application of Bochner's Theorem that a symmetric function $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$ is positive definite, if and only if

$$\varphi(n) = \int_0^\pi \cos n\theta \, d\mu(\theta), \quad n \in \mathbb{Z}, \quad (1.1)$$

for a positive, finite Borel measure μ on $[0, \pi]$. Moreover, μ is uniquely determined by φ , and $\varphi(0) = \mu([0, \pi])$. Similarly, it is a simple consequence of Lévy-Khinchin's formula for real valued functions in the form of [2, Corollary 18.20] that a symmetric function $\psi : \mathbb{Z} \rightarrow \mathbb{C}$ with $\psi(0) = 0$ is conditionally negative definite, if and only if

$$\psi(n) = cn^2 + \int_0^\pi (1 - \cos n\theta) \, d\rho(\theta), \quad n \in \mathbb{Z}, \quad (1.2)$$

for a constant $c \geq 0$ and a Borel measure ρ on $]0, \pi]$ for which

$$\int_0^\pi (1 - \cos \theta) \, d\rho(\theta) < \infty.$$

Moreover, c and ρ are uniquely determined by ψ . If we put

$$\nu = c\delta_0 + (1 - \cos \theta) \, d\rho(\theta),$$

then ν is a finite Borel measure on $[0, \pi]$ uniquely determined by ψ , and

$$\psi(n) = \int_0^\pi \frac{1 - \cos n\theta}{1 - \cos \theta} \, d\nu(\theta), \quad n \in \mathbb{Z}, \quad (1.3)$$

where the integrand for $\theta = 0$ should be replaced by

$$n^2 = \lim_{\theta \rightarrow 0} \frac{1 - \cos n\theta}{1 - \cos \theta}.$$

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Moreover, $\psi(1) = \nu([0, \pi])$. Changing the variable θ to $s = \cos \theta$, the equations (1.1) and (1.3) can be rewritten as

$$\varphi(n) = \int_{-1}^1 T_n(s) d\mu'(s), \quad n \geq 0, \quad (1.4)$$

and

$$\psi(n) = \int_{-1}^1 \frac{1 - T_n(s)}{1 - s} d\nu'(s), \quad n \geq 0, \quad (1.5)$$

where μ' and ν' are the image measures of μ and ν under the map $\theta \mapsto \cos \theta$, $(T_n)_{n=0}^\infty$ are the Chebyshev polynomials of first kind, and where the integrand in (1.5) for $s = 1$ should be replaced by

$$\lim_{s \rightarrow 1} \frac{1 - T_n(s)}{1 - s} = T'_n(1) = n^2.$$

In this paper we will prove formulas analogous to (1.4) and (1.5) for radial functions on free groups of two or more generators, i.e. functions on \mathbb{F}_r ($2 \leq r \leq \infty$) which only depend on the word length $|x|$ of an element $x \in \mathbb{F}_r$.

In [5], Figà-Talamanca and Picardello introduced the notion of spherical functions on the free groups \mathbb{F}_r ($2 \leq r < \infty$). They are complex, radial functions φ on \mathbb{F}_r , i.e. of the form

$$\varphi(x) = \dot{\varphi}(|x|), \quad x \in \mathbb{F}_r,$$

for a unique function $\dot{\varphi} : \mathbb{N}_0 \rightarrow \mathbb{C}$, originally indexed by a parameter $z \in \mathbb{C}$. In this paper we will use $s = \dot{\varphi}(1) = \frac{q}{q+1}(q^{-z} + q^{1-z})$ as the parameter, where $q = 2r - 1$. We will show in Section 2 that with this parametrization the spherical functions $(\varphi_s)_{s \in \mathbb{C}}$ are given by

$$\dot{\varphi}_s(n) = \left[\frac{2}{q+1} T_n \left(\frac{q+1}{2\sqrt{q}} s \right) + \frac{q-1}{q+1} U_n \left(\frac{q+1}{2\sqrt{q}} s \right) \right] q^{-n/2}, \quad n \in \mathbb{N}_0,$$

where $(T_n)_{n=0}^\infty$ and $(U_n)_{n=0}^\infty$ are the Chebyshev polynomials of first and second kind, respectively. For $r = \infty$ we follow the convention of [7] and define the spherical functions on \mathbb{F}_∞ by $\varphi_s(x) = s^{|x|}$. We then prove in Section 3 and 4 the following results analogous to (1.4) and (1.5).

Theorem 1.1. *Let $2 \leq r \leq \infty$ and let $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ be a radial function. The following are equivalent.*

- (1) *The function φ is positive definite.*
- (2) *There is a finite positive Borel measure μ on $[-1, 1]$ such that*

$$\varphi(x) = \int_{-1}^1 \varphi_s(x) d\mu(s), \quad x \in \mathbb{F}_r.$$

Moreover, if (2) holds, then μ is uniquely determined by φ , and $\varphi(e) = \mu([-1, 1])$.

Theorem 1.2. *Let $2 \leq r \leq \infty$ and let $\psi : \mathbb{F}_r \rightarrow \mathbb{C}$ be a radial function with $\psi(e) = 0$. The following are equivalent.*

- (1) *The function ψ is conditionally negative definite.*
- (2) *There is a finite positive Borel measure ν on $[-1, 1]$ such that*

$$\psi(x) = \int_{-1}^1 \psi_s(x) d\nu(s), \quad x \in \mathbb{F}_r,$$

where

$$\psi_s(x) = \frac{1 - \varphi_s(x)}{1 - s}, \quad s \in \mathbb{C} \setminus \{1\},$$

and

$$\psi_1(x) = \lim_{s \rightarrow 1} \frac{1 - \varphi_s(x)}{1 - s}.$$

Moreover, if (2) holds, then ν is uniquely determined by ψ , and $\nu([-1, 1]) = \psi(x)$, when $|x| = 1$.

Note that (1.4) and (1.5) can be considered as the case $q = r = 1$ of Theorem 1.1 and Theorem 1.2.

In Proposition 5.4 we show that $\psi_s(x) \leq a|x|$ for some constant $a \geq 0$ which is independent of s . Thus, as a corollary of Theorem 1.2 we obtain that every radial, conditionally negative definite map $\psi : \mathbb{F}_r \rightarrow \mathbb{C}$ with $\psi(e) = 0$ satisfies the linear bound

$$\psi(x) \leq c|x|, \quad x \in \mathbb{F}_r \quad (1.6)$$

for some constant $c \geq 0$. We note that ψ is necessarily non-negative.

The paper is organized as follows. In Section 2 we recall some facts about spherical functions on free groups. Section 3 and 4 contains the integral representation theorems of positive definite and conditionally negative definite radial functions, respectively. Lastly, in Section 5 we apply the integral representation to deduce the linear bound on the growth of radial, conditionally negative definite functions. The linear bound (1.6) is actually a special case of Theorem 1.7 from [8], but the proof given here of our special case is more illuminating (and much shorter).

2. SPHERICAL FUNCTIONS

Let r be a natural number with $r \geq 2$, and consider the free group \mathbb{F}_r with r generators. To ease notation in some places we let $q = 2r - 1$ (the same notation is used in [4], where it is also better justified). If we identify \mathbb{F}_r with the vertices of its Cayley graph, then each non-trivial element $x \in \mathbb{F}_r$ has q neighbors further from the identity and a single neighbor closer to the identity. The Cayley graph is a homogeneous tree of degree $q + 1$, and the length function on \mathbb{F}_r simply returns the distance to the neutral element. A function $\mathbb{F}_r \rightarrow \mathbb{C}$ is called *radial*, if its value at $x \in \mathbb{F}_r$ only depends on the word length $|x|$. In other words, a function $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ is a radial, if and only if it has the form $\varphi(x) = \dot{\varphi}(|x|)$ for a (necessarily unique) unique map $\dot{\varphi} : \mathbb{N}_0 \rightarrow \mathbb{C}$.

For each $n \in \mathbb{N}_0$ we let $E_n = \{x \in \mathbb{F}_r \mid |x| = n\}$. A simple counting argument (the one given above) shows that $|E_n| = (q + 1)q^{n-1}$, when $n \geq 1$. Let $\mu_0 = \delta_e$ be the Dirac function at e , and let μ_n be the function on \mathbb{F}_r with value $1/(q + 1)q^{n-1}$ on words of length n and zero otherwise.

Let $\mathbb{C}[\mathbb{F}_r]$ denote the group algebra (with convolution as product), and let $\mathcal{A} \subseteq \mathbb{C}[\mathbb{F}_r]$ denote the subspace consisting of (finitely supported) radial functions. Clearly \mathcal{A} is the linear span of $(\mu_n)_{n=0}^\infty$. We call \mathcal{A} the radial algebra. The following is well-known (see [5]).

Lemma 2.1. *For $n \geq 1$ we have*

$$\mu_1 * \mu_n = \frac{1}{q + 1} \mu_{n-1} + \frac{q}{q + 1} \mu_{n+1}, \quad (2.1)$$

and thus the radial algebra \mathcal{A} is the unital, commutative convolution algebra generated μ_1 .

If we let P_n be the polynomials defined recursively by

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x, \\ P_{n+1}(x) &= \frac{q+1}{q} x P_n(x) - \frac{1}{q} P_{n-1}(x), \end{aligned} \tag{2.2}$$

then it is immediate from the lemma that

$$\mu_n = P_n(\mu_1), \quad n \in \mathbb{N}_0. \tag{2.3}$$

Whenever $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ is a function, we let $L\varphi(x)$ denote the average value of φ over the neighbours of x . A computation shows that $(\mu_1 * \varphi)(x)$ is $L\varphi(x)$, so the operator L is simply left convolution by μ_1 . The operator L is called the *Laplace operator*.

We define spherical functions as the eigenfunctions of the Laplace operator.

Definition 2.2. With \mathbb{F}_r the free group on r generators ($2 \leq r < \infty$) we say that a map $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ is *spherical*, if the following two conditions hold.

- (i) φ is radial with $\varphi(e) = 1$,
- (ii) $L\varphi = s\varphi$ for some $s \in \mathbb{C}$.

The number $s \in \mathbb{C}$ is called the *eigenvalue* of the spherical function φ .

If we (as always) let $\dot{\varphi} : \mathbb{N}_0 \rightarrow \mathbb{C}$ denote the unique map such that $\varphi(x) = \dot{\varphi}(|x|)$, then one can rewrite (i) and (ii) as

$$\begin{aligned} \dot{\varphi}(0) &= 1, & \dot{\varphi}(1) &= s, \\ \dot{\varphi}(n+1) &= \frac{q+1}{q} s \dot{\varphi}(n) - \frac{1}{q} \dot{\varphi}(n-1). \end{aligned} \tag{2.4}$$

It follows that to each $s \in \mathbb{C}$ there is a unique spherical function with eigenvalue s , and the eigenvalue is determined by the function's value on words of length one. We denote this function φ_s . Note that in [4], [5] and [7] our φ_s is denoted ϕ_z , where $z \in \mathbb{C}$ can be any complex number satisfying $s = \frac{q}{q+1}(q^{-z} + q^{z-1})$. From (2.2) and (2.4) we immediately see that

$$\dot{\varphi}_s(n) = P_n(s). \tag{2.5}$$

for each $n \in \mathbb{N}_0$ and $s \in \mathbb{C}$.

Recall that the Chebyshev polynomials of first and second kind are defined recursively by

$$\begin{aligned} T_0(x) &= 1, & U_0(x) &= 1, \\ T_1(x) &= x, & U_1(x) &= 2x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), & U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x). \end{aligned}$$

Using the recurrence relation (2.2) one can easily show that P_n can be expressed using Chebyshev polynomials. Explicitly, the following holds.

$$P_n(x) = \left[\frac{2}{q+1} T_n\left(\frac{q+1}{2\sqrt{q}} x\right) + \frac{q-1}{q+1} U_n\left(\frac{q+1}{2\sqrt{q}} x\right) \right] q^{-n/2}, \quad x \in \mathbb{C} \tag{2.6}$$

for each $n \in \mathbb{N}_0$.

When $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ and $\psi \in \mathcal{A}$ we let

$$\langle \psi, \varphi \rangle = \sum_{x \in \mathbb{F}_r} \psi(x) \varphi(x).$$

Observe that when φ is radial, then

$$\langle \mu_n, \varphi \rangle = \dot{\varphi}(n).$$

In particular, (2.5) shows that

$$\langle \mu_n, \varphi_s \rangle = P_n(s). \quad (2.7)$$

We have the following alternative characterization of spherical functions (from [5]).

Lemma 2.3 ([5, Lemma 2]). *Let $2 \leq r < \infty$, and let $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ be a radial map, not identically zero. Then the following are equivalent.*

- (1) *The map φ is spherical.*
- (2) *The functional $f : \psi \mapsto \langle \psi, \varphi \rangle$ is multiplicative on the radial algebra \mathcal{A} .*

Proof. Suppose first φ is spherical with eigenvalue s . Then from (2.3) and (2.7) we see that

$$\langle P_n(\mu_1), \varphi_s \rangle = P_n(s),$$

and since $\{P_n\}_{n=0}^\infty$ spans the set of all polynomials, we get

$$\langle P(\mu_1), \varphi_s \rangle = P(s), \quad \text{for every } P \in \mathbb{C}[x].$$

Since μ_1 generates \mathcal{A} , this shows that f is multiplicative on \mathcal{A} .

Suppose conversely that f is multiplicative on \mathcal{A} . We will show that φ satisfies (2.4) for some $s \in \mathbb{C}$. Since μ_0 is the unit of \mathcal{A} , we have

$$\langle \mu_n, \varphi \rangle = \langle \mu_0, \varphi \rangle \langle \mu_n, \varphi \rangle,$$

and because φ is not identically zero, we must have $\dot{\varphi}(0) = \langle \mu_0, \varphi \rangle = 1$.

Let $s = \dot{\varphi}(1) = \langle \mu_1, \varphi \rangle$. We have

$$\langle \mu_1 * \mu_n, \varphi \rangle = \langle \mu_1, \varphi \rangle \langle \mu_n, \varphi \rangle = s \dot{\varphi}(n).$$

Also by (2.1) we have

$$\langle \mu_1 * \mu_n, \varphi \rangle = \frac{1}{q+1} \langle \mu_{n-1}, \varphi \rangle + \frac{q}{q+1} \langle \mu_{n+1}, \varphi \rangle = \frac{1}{q+1} \dot{\varphi}(n-1) + \frac{q}{q+1} \dot{\varphi}(n+1).$$

This proves (2.4) and shows that φ is spherical. \square

To define the spherical functions on \mathbb{F}_∞ , we first define the Laplace operator L on *radial* functions on \mathbb{F}_∞ . When φ is a radial function on \mathbb{F}_∞ we let $L\varphi$ denote the function on \mathbb{F}_∞ given by

$$L\varphi(x) = \dot{\varphi}(|x| + 1), \quad x \in \mathbb{F}_\infty.$$

Definition 2.4. As before, we say that a radial function $\varphi : \mathbb{F}_\infty \rightarrow \mathbb{C}$ with $\varphi(e) = 1$ is *spherical*, if $L\varphi = s\varphi$ for some $s \in \mathbb{C}$ called the *eigenvalue*.

We see that a radial function φ is spherical with eigenvalue s , if and only if it satisfies the following recurrence relation.

$$\begin{aligned} \dot{\varphi}(0) &= 1, & \dot{\varphi}(1) &= s, \\ \dot{\varphi}(n+1) &= s \dot{\varphi}(n). \end{aligned} \quad (2.8)$$

Obviously, the spherical function with eigenvalue $s \in \mathbb{C}$ is given by

$$\dot{\varphi}(n) = s^n, \quad n \in \mathbb{N}_0.$$

We will need to work with spherical functions on several free groups at once, and when this is the case we denote the spherical function on \mathbb{F}_r with eigenvalue s by $\varphi_{s,r}$. Comparing (2.4) with (2.8) we see that for each $s \in \mathbb{C}$ and $n \in \mathbb{N}_0$

$$\dot{\varphi}_{s,r}(n) \rightarrow \dot{\varphi}_{s,\infty}(n) \quad \text{as } r \rightarrow \infty. \quad (2.9)$$

3. POSITIVE DEFINITE MAPS

In this section we prove Theorem 1.1 first in the case where $2 \leq r < \infty$, and then from this we deduce the case $r = \infty$ by a limiting argument. Recall that a map $\varphi : G \rightarrow \mathbb{C}$ defined on a group G is called positive definite, if

$$\sum_{j,k=1}^n c_j \overline{c_k} \varphi(x_k^{-1} x_j) \geq 0$$

for every $n \in \mathbb{N}$, $\{x_1, \dots, x_n\} \subseteq G$ and $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$.

In [4, p. 53] it is shown that the spherical function φ_s is positive definite, if and only if $s \in [-1, 1]$. This can also be seen in a different way using the following fact, which is a consequence of The Haagerup-Paulsen-Wittstock Factorization Theorem for completely bounded maps [3, Theorem B.7]: For a map $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ with $\varphi(e) = 1$, it holds that φ is positive definite, if and only if the Herz-Schur norm of φ is 1. In [7, Theorem 3.3] the authors calculate the Herz-Schur norm of spherical functions, and it follows from their theorem that φ_s is positive definite, if and only if $s \in [-1, 1]$.

Theorem 3.1. *Let $r \geq 2$ be a natural number, and let $\varphi : \mathbb{F}_r \rightarrow \mathbb{C}$ be radial with $\varphi(e) = 1$. The following are equivalent.*

- (1) *The map φ is positive definite.*
- (2) *There is a probability measure μ on $[-1, 1]$ such that*

$$\varphi(x) = \int_{-1}^1 \varphi_s(x) d\mu(s), \quad x \in \mathbb{F}_r, \quad (3.1)$$

where φ_s is the spherical function given by (2.6).

Moreover, if (2) holds, then μ is uniquely determined by φ .

Proof.

(2) \implies (1). This is trivial, since each φ_s is positive definite.

(1) \implies (2). Consider the universal C^* -algebra $C^*(\mathbb{F}_r)$. Every state on $C^*(\mathbb{F}_r)$ restricts to a positive definite map \mathbb{F}_r , and by universality every positive definite map ρ on \mathbb{F}_r with $\rho(e) = 1$ extends by linearity and continuity to a state on $C^*(\mathbb{F}_r)$. Let Φ be the extension of φ to $C^*(\mathbb{F}_r)$ so that $\Phi(x) = \varphi(x)$ for every $x \in \mathbb{F}_r \subseteq C^*(\mathbb{F}_r)$. We also let Φ_s denote the extension of φ_s to a state on $C^*(\mathbb{F}_r)$, when $s \in [-1, 1]$.

Let $(\mu_n)_{n=0}^\infty$ be as above, and consider them as elements of $C^*(\mathbb{F}_r)$ in the usual way. The element μ_1 generates the radial C^* -subalgebra $C^*(\mu_1)$ of $C^*(\mathbb{F}_r)$. By Lemma 2.3 each Φ_s is multiplicative on $C^*(\mu_1)$. Since $\mu_1 = \mu_1^*$ and $\|\mu_1\| \leq 1$, the spectrum $\sigma(\mu_1)$ of μ_1 is a subset of $[-1, 1]$. Conversely, by (2.7) we see that $\Phi_s(\mu_1) = s$, so $[-1, 1] \subseteq \sigma(\mu_1)$.

By spectral theory $C^*(\mu_1) \simeq C([-1, 1])$. Restricting Φ to $C^*(\mu_1)$ gives a state which by the Riesz Representation Theorem has the form

$$\Phi(f(\mu_1)) = \int_{-1}^1 f \, d\mu, \quad f \in C([-1, 1]) \quad (3.2)$$

for a unique probability measure μ on $[-1, 1]$. Since φ is radial,

$$\dot{\varphi}(n) = \Phi(\mu_n) = \Phi(P_n(\mu_1))$$

and we know from (3.2) and (2.5) that

$$\Phi(P_n(\mu_1)) = \int_{-1}^1 P_n(s) \, d\mu(s) = \int_{-1}^1 \dot{\varphi}_s(n) \, d\mu(s).$$

This shows

$$\dot{\varphi}(n) = \int_{-1}^1 \dot{\varphi}_s(n) \, d\mu(s), \quad n = 0, 1, 2, \dots \quad (3.3)$$

We now turn to prove uniqueness. Note that if μ is a probability measure satisfying (3.1) then we must have

$$\Phi(P_n(\mu_1)) = \Phi(\mu_n) = \dot{\varphi}(n) = \int_{-1}^1 \dot{\varphi}_s(n) \, d\mu(s) = \int_{-1}^1 P_n(s) \, d\mu(s),$$

so it must be the unique measure guaranteed by the Riesz Representation Theorem, since the polynomials $\{P_n\}_{n=0}^\infty$ span the set of all polynomials. \square

We wish to prove an analogue of the previous theorem for positive definite, radial maps on \mathbb{F}_∞ . Since we no longer have the aid of the radial algebra, the idea is instead to use the previous theorem together with a limit argument. But first we identify those spherical functions on \mathbb{F}_∞ which are positive definite.

Since $\varphi_{s,r}$ is positive definite, when $s \in [-1, 1]$, we get by (2.9) that $\varphi_{s,\infty}$ is positive definite, when $s \in [-1, 1]$. This may also be seen directly as follows.

When $s \in \{0, 1\}$, this is easy. When $0 < s < 1$, it is a well-known result from [6] that $s \mapsto s^{|x|}$ is positive definite. When $-1 \leq s < 0$ we write $s^{|x|} = (-1)^{|x|}(-s)^{|x|}$, and since a product of positive definite maps is again positive definite, it now suffices to show that $x \mapsto (-1)^{|x|}$ is positive definite on \mathbb{F}_∞ .

Notice that the parity of $|xy|$ and $|x| + |y|$ is the same, since the reduced form of xy is obtained by cancelling the same number of letters from x and y . Hence $(-1)^{|xy|} = (-1)^{|x|+|y|}$, so $x \mapsto (-1)^{|x|}$ is a homomorphism of \mathbb{F}_∞ into the unit circle in \mathbb{C} . It is easily seen that such homomorphisms are always positive definite.

Positive definite, radial functions are necessarily real and bounded, so when $s \notin [-1, 1]$, the map $\varphi_{s,\infty}$ cannot be positive definite. This shows that the positive definite spherical functions on \mathbb{F}_∞ are precisely those with eigenvalue $s \in [-1, 1]$.

Lemma 3.2. *For each $x \in \mathbb{F}_r$ the functions $s \mapsto \varphi_{s,r}(x)$ converge uniformly to $s \mapsto \varphi_{s,\infty}(x)$ as $r \rightarrow \infty$.*

Proof. We will estimate the value

$$\delta(r, n) = \sup_{s \in [-1, 1]} |\dot{\varphi}_{s,r}(n) - \dot{\varphi}_{s,\infty}(n)|.$$

When $n \in \{0, 1\}$, we obviously have $\delta(r, n) = 0$. When $n \geq 1$ we find using the recurrence relations (2.4) and (2.8)

$$|\dot{\varphi}_{s,r}(n+1) - \dot{\varphi}_{s,\infty}(n+1)| \leq |\dot{\varphi}_{s,r}(n) - \dot{\varphi}_{s,\infty}(n)| + \frac{2}{2r-1},$$

so

$$\delta(r, n+1) \leq \delta(r, n) + \frac{2}{2r-1},$$

By induction over n it follows that $\delta(r, n) \rightarrow 0$ as $r \rightarrow \infty$ for each $n \in \mathbb{N}_0$. \square

We are now ready to prove a version of Theorem 3.1, when r is infinite.

Theorem 3.3. *Let $\varphi : \mathbb{F}_\infty \rightarrow \mathbb{C}$ be radial with $\varphi(e) = 1$. The following are equivalent.*

- (1) *The map φ is positive definite.*
- (2) *There is a probability measure μ on $[-1, 1]$ such that*

$$\dot{\varphi}(n) = \int_{-1}^1 s^n d\mu(s), \quad n \in \mathbb{N}. \quad (3.4)$$

Moreover, if (2) holds, then μ is uniquely determined by φ .

Proof.

(2) \implies (1). We have seen that the map $x \mapsto s^{|x|}$ is positive definite on \mathbb{F}_∞ for each $s \in [-1, 1]$. Hence φ is positive definite.

(1) \implies (2). We consider the chain of subgroups $\mathbb{F}_2 \subseteq \mathbb{F}_3 \subseteq \dots \subseteq \mathbb{F}_\infty$. Let $\varphi_{s,r}$ denote the spherical function on \mathbb{F}_r with parameter $s \in \mathbb{C}$.

Suppose $\varphi : \mathbb{F}_\infty \rightarrow \mathbb{C}$ is positive definite, radial and satisfies $\varphi(e) = 1$. The restriction of φ to the subgroup \mathbb{F}_r is still radial and positive definite. Using Theorem 3.1, when r is finite, we find a probability measure μ_r on $[-1, 1]$ such that

$$\dot{\varphi}(n) = \int_{-1}^1 \dot{\varphi}_{s,r}(n) d\mu_r(s) \quad (3.5)$$

for all $n \in \mathbb{N}_0$. Let μ be any cluster point (in the vague topology) of the sequence $(\mu_r)_{r=2}^\infty$. We claim that

$$\dot{\varphi}(n) = \int_{-1}^1 \dot{\varphi}_{s,\infty}(n) d\mu(s), \quad n \in \mathbb{N}_0. \quad (3.6)$$

Let $n \in \mathbb{N}_0$ and $\varepsilon > 0$ be given. Since μ is a cluster point of $(\mu_r)_{r=2}^\infty$, there is some r_0 such that

$$\left| \int_{-1}^1 \dot{\varphi}_{s,\infty}(n) d\mu(s) - \int_{-1}^1 \dot{\varphi}_{s,\infty}(n) d\mu_r(s) \right| < \varepsilon \quad (3.7)$$

for infinitely many $r \geq r_0$. Further, since $\dot{\varphi}_{s,r}(n) \rightarrow \dot{\varphi}_{s,\infty}(n)$ uniformly as $r \rightarrow \infty$, there is some $r \geq r_0$ such that (3.7) holds and

$$\left| \int_{-1}^1 \dot{\varphi}_{s,\infty}(n) d\mu_r(s) - \int_{-1}^1 \dot{\varphi}_{s,r}(n) d\mu_r(s) \right| < \varepsilon.$$

Combining this with (3.5), we obtain

$$\left| \dot{\varphi}(n) - \int_{-1}^1 \dot{\varphi}_{s,\infty}(n) d\mu(s) \right| < 2\varepsilon,$$

which proves (3.6).

Uniqueness follows in the same way as in the proof of Theorem 3.1. \square

Theorem 1.1 is the combination of Theorem 3.1 and Theorem 3.3 modulo scaling of the value $\varphi(e)$.

Remark 3.4. Note that (2) in Theorem 3.3 coincides with the characterization of positive definite functions on the semigroup \mathbb{N}_0 given in [1, Proposition 4.9].

4. CONDITIONALLY NEGATIVE DEFINITE MAPS

In this section we introduce a family of conditionally negative definite maps on \mathbb{F}_r parametrized by the interval $[-1, 1]$ and prove that every radial conditionally negative definite map has an integral representation using these.

Recall that a map $\psi : G \rightarrow \mathbb{C}$ defined on a group G is called conditionally negative definite, if ψ is hermitian, i.e. $\psi(x^{-1}) = \overline{\psi(x)}$ when $x \in G$, and

$$\sum_{j,k=1}^n c_j \overline{c_k} \psi(x_k^{-1} x_j) \leq 0$$

for every $n \in \mathbb{N}$, $\{x_1, \dots, x_n\} \subseteq G$ and $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$ such that $\sum_j c_j = 0$. Note that a radial, conditionally negative definite function is real-valued.

When $x \in \mathbb{F}_r$ we define

$$\begin{aligned} \psi_s(x) &= \frac{1 - \varphi_s(x)}{1 - s}, \quad -1 \leq s < 1 \\ \psi_1(x) &= \lim_{s \rightarrow 1} \psi_s(x). \end{aligned} \tag{4.1}$$

Clearly each map ψ_s is radial. Since the map $s \mapsto \dot{\varphi}_s(n)$ is a polynomial in s , it is differentiable at $s = 1$, so the definition of ψ_1 makes sense, and $\dot{\psi}_1(n) = \dot{\varphi}_1'(n)$. Note that $s \mapsto \dot{\psi}_s(n)$ is a polynomial in s of degree $n - 1$.

Each φ_s is positive definite when $s \in [-1, 1]$, so ψ_s is conditionally negative definite for each $s \in [-1, 1[$, and since ψ_1 is the pointwise limit of conditionally negative maps, it is itself conditionally negative definite.

Let $s \in [-1, 1]$. Since φ_s is positive definite and radial, it is real. And also $|\varphi_s(x)| \leq \varphi_s(e) = 1$ for each $x \in \mathbb{F}_r$. In particular, each ψ_s is non-negative with $\psi_s(e) = 0$.

Theorem 4.1. *Let $2 \leq r \leq \infty$, and let $\psi : \mathbb{F}_r \rightarrow \mathbb{C}$ be radial with $\psi(e) = 0$. The following are equivalent.*

- (1) *The map ψ is conditionally negative definite.*
- (2) *There is a finite positive Borel measure ν on $[-1, 1]$ such that*

$$\psi(x) = \int_{-1}^1 \psi_s(x) d\nu(s), \quad x \in \mathbb{F}_r,$$

where ψ_s is the function given by (4.1).

Moreover, if (2) holds, then ν is uniquely determined by ψ , and $\nu([-1, 1]) = \psi(x)$, when $|x| = 1$.

Proof.

(2) \implies (1). This is trivial, since each ψ_s is conditionally negative definite.

(1) \implies (2). For $t > 0$ the map $e^{-t\psi}$ is radial, positive definite with $e^{-t\psi(e)} = 1$, so there is a probability measure μ_t on $[-1, 1]$ such that

$$e^{-t\psi(x)} = \int_{-1}^1 \varphi_s(x) d\mu_t(s), \quad x \in \mathbb{F}_r.$$

Let ν_t be the positive measure with density $s \mapsto (1-s)/t$ with respect to μ_t . Then

$$\frac{1 - e^{-t\psi(x)}}{t} = \int_{-1}^1 \frac{1 - \varphi_s(x)}{t} d\mu_t(s) = \int_{-1}^1 \psi_s(x) d\nu_t(s).$$

To see the last equality, observe that both integrals may just as well be taken over the half-open interval $[-1, 1[$, since

$$\frac{1 - \varphi_1(x)}{t} = 0 \quad \text{and} \quad \nu_t(\{1\}) = 0.$$

When $|x| = 1$, we have

$$\nu_t([-1, 1]) = \frac{1 - e^{-t\psi(x)}}{t} \nearrow \psi(x) \quad \text{as } t \rightarrow 0.$$

Hence ν_t lies in a bounded subset of $M^+([-1, 1])$, the space of positive Radon measures on $[-1, 1]$. By compactness there is a subnet (ν_{t_α}) that converges vaguely to, say ν . Now,

$$\int_{-1}^1 \psi_s(x) d\nu = \lim_{\alpha} \int_{-1}^1 \psi_s(x) d\nu_{t_\alpha} = \lim_{\alpha} \frac{1 - e^{-t_\alpha\psi(x)}}{t_\alpha} = \lim_{t \rightarrow 0} \frac{1 - e^{-t\psi(x)}}{t} = \psi(x)$$

as desired.

Uniqueness of ν follows as usual, since $\psi_s(n)$ is a polynomial in s of degree $n - 1$. \square

5. LINEAR BOUND

In this section we apply the integral representation in Theorem 4.1 to give a bound on the growth of conditionally negative definite, radial functions on the free groups. The result is contained in Corollary 5.5.

The following result about the Chebyshev polynomials will be relevant to us.

Lemma 5.1. *Let P be a Chebyshev polynomial of either kind, and let $x_0 > 1$. For any $x \in \mathbb{R}$ such that $|x| < x_0$ we have*

$$\frac{P(x_0) - P(x)}{x_0 - x} \leq P'(x_0). \quad (5.1)$$

Proof. A glance at the mean value theorem shows that it suffices to prove that $P'(x) \leq P'(x_0)$ for every x with $|x| < x_0$. Recall that Chebyshev polynomials are Jacobi polynomials, and since P is a Jacobi polynomial, so is P' up to a positive scaling factor (see (4.21.7) in [9]). It is known (see p. 168 in [9]) that $P'(x)$ restricted to the interval $[-1, 1]$ attains its maximum at the end-point $x = 1$. We know that P' is either even or odd. Finally, it is well-known that Jacobi polynomials are increasing on $[1, \infty[$, so $P'(x) \leq P'(x_0)$, when $x \in]-x_0, x_0[$. \square

Lemma 5.2. *Let $2 \leq r \leq \infty$. For each $s \in [-1, 1]$ and $x \in \mathbb{F}_r$ we have*

$$\psi_s(x) \leq \psi_1(x).$$

Proof. We may of course assume that $s \neq 1$. If $r = \infty$, the result is obvious, since then $\psi_s(x) = 1 + s + s^2 + \dots + s^{|x|-1}$. So we suppose $2 \leq r < \infty$. Recall that then $\dot{\varphi}_s(n) = P_n(s)$, where P_n has the form given in (2.6), and that

$$\dot{\psi}_s(n) = \frac{1 - P_n(s)}{1 - s}.$$

Thus, we must prove that the slope of the secant line of P_n through the points $(1, P_n(1))$ and $(s, P_n(s))$ is bounded above by the slope of the tangent line of P_n at 1. To this end, let

$$t_n(s) = T_n\left(\frac{q+1}{2\sqrt{q}}s\right)q^{-n/2}, \quad u_n(s) = U_n\left(\frac{q+1}{2\sqrt{q}}s\right)q^{-n/2},$$

and put $\lambda = 2/(q+1)$, so that

$$P_n(s) = \lambda t_n(s) + (1 - \lambda)u_n(s).$$

It suffices to show that

$$\frac{t_n(1) - t_n(s)}{1 - s} \leq t'_n(s) \quad \text{and} \quad \frac{u_n(1) - u_n(s)}{1 - s} \leq u'_n(s),$$

when $-1 \leq s < 1$. This, in turn, is equivalent to showing

$$\frac{T_n(s_0) - T_n(s)}{s_0 - s} \leq T'_n(s_0) \quad \text{and} \quad \frac{U_n(s_0) - U_n(s)}{s_0 - s} \leq U'_n(s_0),$$

when $-s_0 \leq s < s_0$ and $s_0 = \frac{q+1}{2\sqrt{q}}$. An application of the previous lemma now completes our proof, since $s_0 > 1$. \square

Recall the following relations concerning hyperbolic functions and Chebyshev polynomials.

$$T_n(\cosh \alpha) = \cosh(n\alpha), \quad U_n(\cosh \alpha) = \frac{\sinh((n+1)\alpha)}{\sinh \alpha} \quad (5.2)$$

for all $\alpha \neq 0$ and $n \in \mathbb{N}_0$.

Lemma 5.3. *Let $2 \leq r < \infty$. The map ψ_1 has the following form.*

$$\dot{\psi}_1(n) = n \frac{q+1}{q-1} - \frac{2q(1-q^{-n})}{(q-1)^2}, \quad n \in \mathbb{N}_0,$$

where as usual $q = 2r - 1$.

Proof. If we let $\alpha = \frac{1}{2} \log q$, then notice that

$$\frac{q+1}{2\sqrt{q}} = \cosh \alpha, \quad \frac{q-1}{q+1} = \tanh \alpha \quad \text{and} \quad \frac{2}{q+1} = \frac{e^{-\alpha}}{\cosh \alpha}.$$

Then (2.6) takes the form

$$P_n(s) = \left[\frac{e^{-\alpha}}{\cosh \alpha} T_n(\cosh(\alpha)s) + \tanh(\alpha) U_n(\cosh(\alpha)s) \right] e^{-\alpha n}.$$

Recall that $\dot{\psi}_1(n) = P'_n(1)$. Using the well-known facts that

$$T'_n(x) = nU_{n-1}(x), \quad U'_n(x) = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2 - 1},$$

we find the following expression for $P'_n(1)$.

$$P'_n(1) = \left[e^{-\alpha} n U_{n-1}(\cosh \alpha) + \sinh \alpha \frac{(n+1)T_{n+1}(\cosh \alpha) - \cosh \alpha U_n(\cosh \alpha)}{\cosh^2 \alpha - 1} \right] e^{-\alpha n}.$$

Using (5.2) we arrive after some reduction at the expression

$$\dot{\psi}_1(n) = \frac{e^{-\alpha n}}{\sinh \alpha} (n e^{-\alpha} \sinh(n\alpha) + (n+1) \cosh((n+1)\alpha) - \coth(\alpha) \sinh((n+1)\alpha)).$$

Rewriting in terms of q gives, again after some reduction,

$$\dot{\psi}_1(n) = n \frac{q+1}{q-1} - \frac{2q(1-q^{-n})}{(q-1)^2}.$$

□

Proposition 5.4. *Let $2 \leq r \leq \infty$. There exists a constant $a \geq 0$ such that*

$$\psi_s(x) \leq a|x|$$

for every $x \in \mathbb{F}_r$ and $s \in [-1, 1]$. In fact, we may take $a = \frac{r}{r-1}$, when r is finite, and we may take $a = 1$ when $r = \infty$.

Proof. By Lemma 5.2 it suffices to find $a \geq 0$ such that $\psi_1(x) \leq a|x|$ for every $x \in \mathbb{F}_r$. It follows from Lemma 5.3 that we may take $a = \frac{q+1}{q-1} = \frac{r}{r-1}$, when $r < \infty$. When $r = \infty$ we have $\psi_1(x) = |x|$, so $a = 1$ clearly works. □

As a corollary of Theorem 4.1 and the previous proposition we obtain the following linear bound on conditionally negative definite, radial maps.

Corollary 5.5. *Let $2 \leq r \leq \infty$, and let $\psi : \mathbb{F}_r \rightarrow \mathbb{R}$ be a radial function. Assume ψ is conditionally negative definite with $\psi(e) = 0$. Then there exists a constant $c \geq 0$ such that $\psi(x) \leq c|x|$ for every $x \in \mathbb{F}_r$.*

Proof. Since ψ has an integral representation as in Theorem 4.1, it follows from Proposition 5.4 that there is a constant $c \geq 0$ such that $\psi(x) \leq c|x|$ for every $x \in \mathbb{F}_r$. In fact, we may take $c = \dot{\psi}(1)a$, where a is the constant in Proposition 5.4. □

Remark 5.6. In Corollary 5.5 the restriction to radial maps is essential. For instance, on the free group \mathbb{F}_∞ there exist conditionally negative definite maps that are unbounded on the set of generators, so clearly they do not admit a bound as in Corollary 5.5. We end by giving such an example.

Let $\psi : \mathbb{Z} \rightarrow \mathbb{R}$ be given by $\psi(n) = n^2$. If $\sum_j c_j = 0$, then

$$\sum_{j,k=1}^n c_j \overline{c_k} (x_j - x_k)^2 = -2 \left| \sum_{j=1}^n c_j x_j \right|^2 \leq 0, \quad x_1, \dots, x_n \in \mathbb{R},$$

so ψ is conditionally negative definite on \mathbb{Z} . Let $\{b_n\}_n$ be the generators of \mathbb{F}_r , and let $\rho : \mathbb{F}_r \rightarrow \mathbb{Z}$ be the homomorphism given by $\rho(b_n) = n$. The composition $\psi \circ \rho$ is then conditionally negative definite on \mathbb{F}_r , and

$$(\psi \circ \rho)(b_n) = n^2$$

Clearly, $\psi \circ \rho$ is not radial. It is also obvious that there is no $c \geq 0$ such that $(\psi \circ \rho)(x) \leq c|x|$ for every $x \in \mathbb{F}_r$, because $(\psi \circ \rho)(b_1^n) = n^2$ for all $n \in \mathbb{N}$, while $|b_1^n| = n$. If $r = \infty$, then $\psi \circ \rho$ is even unbounded on the set of generators.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN,
 UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN Ø, DENMARK
E-mail address: haagerup@math.ku.dk

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN,
 UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN Ø, DENMARK
E-mail address: knudby@math.ku.dk